

BOOK REVIEWS

Mathematical Topics in Fluid Mechanics, Volume 1, Incompressible Models, Oxford Lecture Series in Mathematics and its Applications—3, by **P. L. Lions** [Clarendon Press (Oxford University Press), Oxford, 1996, 237 pp.] GB£ 23.50 hc ISBN 0 19 851487 5

As announced in the title this is the first volume, of a series of two, devoted to some mathematical questions raised by the equations of Fluid Mechanics. The equations of macroscopic fluid mechanics are well established. Their derivation from first principles (recalled in the introduction) is very well understood. However the mathematical treatment of these equations is far from being complete and it has raised a series of very difficult questions and stimulated much progress in mathematics. For instance one may notice that precursors of so-called “distributions” (or generalized functions) are present in the solution of the Riemann problem for gas dynamics and in the so called “turbulent solutions” introduced by Leray.

In a mathematical approach the first natural step is the proof of the existence and uniqueness or stability of the solutions. The book is mostly devoted to these considerations. The examination of such questions is more than an intellectual game for mathematicians. On the one hand it is a way to prove that the equations are well adapted to the description of the problem. On the other, the proof of regularity or uniqueness is related to the amount of information available concerning the appearance of singular behaviour which may itself be linked to turbulence.

Finally the use of large scale computers leads to the discretization of the real fluid dynamical equations and the validity of these discrete computations (with a large, but finite, number of degrees of freedom) is closely related to the stability of the solutions for the continuous problem.

However one should be aware that the gap between what is rigorously proven and what is currently needed for applications is still enormous. Even if constant progress is currently being made, essential questions such as the validity of the Kolmogorov scaling, the derivation of turbulent models (for instance the $k - \varepsilon$ model) or the large-time flow behaviour are not yet fully understood. These problems are mostly based on phenomenological considerations and therefore do not usually appear in a mathematical treatise on fluid mechanics.

Volume 1 is devoted to incompressible fluids while the second will be concerned with compressible fluids.

A series of new results is given with full details included and they are compared with previous classical theory. In some cases complete proofs for these classical results are presented while for others, well documented references can be found in a very large and excellent bibliography.

The introduction is concerned with the (now classical) derivation of the equations of fluid mechanics from first principles and therefore the most general form of the “Compressible Navier-Stokes system” is obtained. From this system the author derives “formally” (without convergence proofs) the validity of the inhomogeneous incompressible model:

$$(1) \quad \partial_t \rho + \nabla_x \cdot (\rho \vec{u}) = 0, \quad \nabla_x \cdot \vec{u} = 0$$

$$(2) \quad \partial_t (\rho \vec{u}) + \nabla_x \cdot (\rho \vec{u} \otimes \vec{u}) - \nabla_x \cdot (2\mu \underline{\underline{d}}) + \nabla_x p = \rho \vec{f}.$$

In (2) μ , which represents the viscosity, is a continuous positive function of the density ρ ,

$$\underline{d} = d_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

represents the deformation tensor and \vec{f} the external forces. Pertinently, the author notices that the above equations are also adapted to the description of some multi-phase flows and in particular to fluid initially surrounded by vacuum. He observes that, in these problems, the regularity of the free boundary is a widely open problem.

Chapter 2 is devoted to system (1)-(2), chapter 3 to the incompressible Navier Stokes equations and the last chapter (4) concerns the incompressible Euler equations (*i.e.* the above equations with zero viscosity). The reader will also find five technical appendices and an extensive bibliography as mentioned above.

For all the problems treated in the book, it is classical and easy to prove the existence and uniqueness of solutions in a class of convenient “regular” functions with the following provisions:

- i) The initial data are assumed to belong to this same class of smooth functions.
- ii) The solution is valid only during a small enough time interval.

The results are closely related to some avatars of the Cauchy-Kowalewsky theorem and the article of Lichtenstein (1930), quoted in the bibliography, is a model of this approach.

However, due to the nonlinearity, the estimates used in the current proof do not remain valid for arbitrarily large times. In the absence of other relevant information they may blow up, as in the solution of the ordinary differential equation

$$y' = y^2.$$

This remark is in full agreement with a very interesting observation made on pages 151-152. Following joint work with R.J. Di Perna the author describes a sequence of solutions $\vec{u}_k(x, t)$ for the incompressible Euler equations in three space variables with the following property: For any $t > 0$ the norm of the solution in Sobolev space $W^{1,p}$ cannot be uniformly (with respect to k) estimated in terms of the same norm as for the initial data $\vec{u}_k(0, x)$.

Finally in many practical (for instance the Kelvin-Helmholtz problem) or numerical (vortex patch method) situations, the initial data are not regular enough to fit in the method based on the Cauchy-Kowalewsky theorem.

The above remarks justify that most of the book is devoted to the derivation of *a priori* estimates (estimates which are obtained without the explicit knowledge of the solution) and to regularity results valid for arbitrarily large times. These constructions are based on physical conservation laws.

There are several good reasons for this approach:

- i) The control of the optimal regularity of the solutions is closely related to the appearance of turbulence.
- ii) These *a priori* estimates are indeed stability estimates and with standard limit processes they will (with the convenient compactness argument) lead to the proof of existence of weak solutions in the spirit of Leray (1934).
- iii) However, very often, these solutions are too “weak” and it is not possible to prove their uniqueness. One thing that could be done is to compare a weak solution with a strong one (if it exists) and, in this setting, aim for a stability result.

Along this line the author introduces for the three-dimensional Euler equations the very neat notion of “dissipative” solution (page 153). This idea may turn out to be also adapted to other problems.

The spirit of the book very well appears in chapter 2 : To prove the existence of solutions of system (1)-(2), it is necessary to go to the limit in expressions of the following type:

$$\rho'' \vec{u}'' \otimes \vec{u}'', \quad \mu(\rho'')(\partial_i u_j'' + \partial_j u_i'')$$

by using only estimates which lead to weak convergence.

To overcome this difficulty, the author first observes that one can go to the limit in the equation

$$\partial_t \rho'' + \nabla_x \cdot (\rho'' \vec{u}'') = 0, \quad \nabla_x \cdot \vec{u}'' = 0,$$

and obtain

$$\partial_t \rho + \nabla_x \cdot (\rho \vec{u}) = 0, \quad \nabla_x \cdot \vec{u} = 0.$$

With the only regularity hypothesis $\vec{u} \in L^2(0, T; H^1(\Omega))$ the analysis of the “weak” solutions of the above evolution equation is not all that easy. In fact this turns out to be the main objective of joint work by the author with R. J. Di Perna. This contribution leads in particular to the relation:

$$(3) \quad \lim_{n \rightarrow \infty} \int_{\Omega} (\rho^n(x, t))^2 dx = \int_{\Omega} (\rho^n(x, 0))^2 dx = \int_{\Omega} (\rho(x, 0))^2 dx = \int_{\Omega} (\rho(x, t))^2 dx.$$

Combined with weak convergence, (3) leads to the strong L^2 convergence which is the essential step in the proof.

In fact with $\rho(x, 0) \equiv 1$ the incompressible Navier-Stokes equations constitute a special case of system (1)-(2). This observation provides a good reason for considering it in Chapter 3.

Some new regularity theorems obtained by the author in collaboration with Meyer and Semmes complete the detailed classical description (in the spirit of Leray).

For instance it is shown that the Leray solution satisfies the relations

$$(4) \quad \partial_{ij} p \in L^1(O, T; \mathcal{H}^1(\mathbf{R}^3))^3 \text{ and } \nabla_x u \in L^\infty(O, T; L^1(\mathbf{R}^3)),$$

where \mathcal{H}^1 denotes the Hardy, the introduction of which comes from the inversion of the Laplacian in the formula

$$\Delta p = \partial_i u_j \partial_j u_i \in L^1(0, T \times \mathbf{R}^3).$$

The second relation in (4) is a consequence of the analysis of the solution to the inhomogeneous heat equation with a right-hand side which is a measure.

The fourth chapter is devoted to the incompressible Euler equations (zero viscosity).

Here also the author starts with a well documented description of the classical results. As mentioned above, an example of instability is given on page 151. This result is simple but of paramount importance for further research in the field. It shows that there is no hope of obtaining new information with the methods of classical functional analysis.

Next the author examines existence theorems in two space variables with singular initial data. In this case, the vorticity is identified with a vector orthogonal to the plane of the fluid motion. Therefore it is conserved along fluid particle trajectories according to the equation

$$(5) \quad \partial_t \omega + \nabla_r \cdot (\omega \vec{u}) = 0.$$

Equation (5) has been systematically used in the past by Wolibner (1933) for the regularity with initial data in the space $C^{1,\alpha}$, and by Yudovich (1955) for the uniqueness of solutions such that

$$\omega(x, 0) \in L^\infty(\Omega).$$

The work of Di Perna and P.-L. Lions, devoted to conservative transport equations with non smooth coefficients, allows to consider initial data that are much less regular. For instance one shows the existence of a solution (with conservation of energy) for initial data which satisfy an Orlicz estimate of the type

$$(6) \quad \int_{\omega} |\omega| \text{Log}|\omega|^{1/2} < \infty.$$

As already mentioned, one of the prime motivations for the introduction of non regular initial data is the Kelvin-Helmholtz problem which correspond to initial data for vorticity defined by a density concentrated on a curve. In this case (less regular than (6)) the existence of a weak solution has been proven for initial data which are measures and which satisfy a “sign” condition:

$$(7) \quad (\omega(x, 0))^- \in L^1(\Omega).$$

However with the hypothesis (7) (instead of (6)) the conservation of energy remains an open problem.

This chapter concludes with some considerations on the Euler equations for incompressible non-homogeneous fluids and on the hydrostatic approximation.

Remarks

Most of the results given in the book are new and if sometimes the proof of classical results is omitted (references are given however) the proof of the new results is presented with full details. The analysis is not easy. Observe for instance that the adaptation to the non-homogeneous case of the classical ideas of Leray (1933) has been the object of a series of contributions ranging over 30 years, and that the optimal result concerning weak solutions, in any space dimension, and with a density dependent viscosity has just recently (1933) been obtained.

As described in the book, the approach relies heavily on the analysis of the weak solution of “conservative” transport equations with singular coefficients (Di Perna and Lions (1983)). In fact this tool appears almost in every chapter and this provide a nice unity of exposition.

In most cases both bounded and unbounded domains are considered with convenient boundary conditions. Special care is given to the proofs in the case of unbounded domains.

In his treatise, the author had to deal with a very large variety of methods in modern analysis (for instance Hardy spaces and solutions of the inhomogeneous heat equation with the right-hand-side being a measure). These methods are described with clarity and precision.

In the introduction, the author claims that “he does not assume from the reader really technical prerequisites other than a basic training in (non linear) partial differential equations”. At variance, one could say that the material presented in the book is not only essential in the mathematical treatment of fluid mechanics but also in many other fields where the theory of non linear partial differential equations plays an important role. Its reading will become a must for researchers in this area of mathematics.

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Scaling, Self-similarity, and Intermediate Asymptotics, Cambridge Texts in Applied Mathematics 14, by **G. I. Barenblatt** (Cambridge University Press, Cambridge 1996, 386 pp.) GB£ 65.00 hc ISBN 0 521 43516 1, GB£ 22.95 pb ISBN 0 521 43522 6

In his book on *Dimensional Analysis* (MacMillan, 1964) Palacios, as an example of the dangers of indiscriminate use of dimensional analysis, asks how much time t a golfer must practice in order to drive a ball a distance d , in earth gravity g , and deduces that $t = C(d/g)^{\frac{1}{2}}$. Batchelor (*Quart. J. Roy. Met. Soc.*, **80** (1954), 339) also notes that dimensional analysis is “...an exceedingly powerful weapon... (but) that it is also a blind, and indiscriminating weapon and that it is fatally easy to prove too much and to get more out of a problem than was put into it”. Of course, there have been dramatic successes, and one of the most spectacular is Taylor’s dimensional analysis of the blast-wave problem which led to the development of a self-similar solution. That this might be described as spectacular arises from the fact that when time series photographs of the first nuclear explosion were published in 1947, Taylor was not only able to confirm his similarity theory, but also make an accurate estimate of the energy released by the bomb. This caused consternation in official circles since that information, from measurements made at the time, was not in the public domain. An interesting historical account of this is given by Batchelor in *The Life and Legacy of G. I. Taylor* (Cambridge, 1996).

One of the aforementioned photographs adorns the front cover of the book under review and, unsurprisingly, the blast-wave problem features prominently in it. In the first chapter the basic ideas of dimensional analysis and similarity are introduced, including the Π theorem, and Taylor’s blast-wave result is given as one example. In the next, on self-similar and intermediate-asymptotic solutions, it again features as one example. The idea of self-similarity, in this case a solution that is valid at times and distances large so that any influence of asymmetry of initial conditions and size of the domain of original energy release is unimportant, is familiar. Barenblatt defines an intermediate-asymptotic solution as one that is self-similar in the above sense, but does not yet represent the final state. As he notes “This situation is common, and greatly increases the significance of self-similar solutions”. The blast-wave similarity solution again provides an example, since it is inappropriate at very large times as it decays to an acoustic wave. A not entirely convincing pictorial image of the *Mona Lisa* is included as an illustration of intermediate asymptotics.

The self-similar solutions that can be constructed using dimensional analysis alone are designated self-similar solutions of the first kind. Others that cannot, where an exponent in the solution must be determined from an eigenvalue problem, are of the second kind. Perhaps the earliest and most famous of the latter is Guderley’s implosion problem. But the present author, as an initial illustration of this second kind, constructs the similarity solution for inviscid, irrotational flow past an infinite wedge as an intermediate-asymptotic solution for flow past a wedge of finite length. Several further examples of this second kind are discussed. Self-similar solutions contain basic constants which for the first kind are determined from integral conservation laws. Although these do not exist for solutions of the second kind there are analogous asymptotic conservation laws discussed for